THE RELATIVE ENERGY FOR ELECTROMAGNETIC FLUIDS

NUNO J. ALVES AND ATHANASIOS E. TZAVARAS

Abstract. This note introduces the relative energy for two-species electromagnetic fluid models. The relative energy identity for a bipolar Euler-Maxwell system is formally derived. Moreover, two applications of the relative energy for these type of systems are given. The first application deals with the weak-strong uniqueness property of a Euler-Poisson system, while the second concerns the relaxation limit of a bipolar Euler-Poisson system towards a bipolar drift-diffusion system.

1. Introduction

In this note, one introduces the relative energy for a bipolar Euler-Maxwell system. The system under consideration is composed of two sets of Euler equations that describe the evolution of a two-species fluid composed of charged particles, e.g. ions and electrons,

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p_1(\rho) = \rho (E + u \times B) \\ \partial_t n + \nabla \cdot (n v) = 0 \\ \partial_t (n v) + \nabla \cdot (n v \otimes v) + \nabla p_2(n) = -n (E + v \times B) \end{cases} \tag{1.1}$$

coupled with Maxwell's equations to account for the electromagnetic effects

$$\begin{cases} \partial_t B + \nabla \times E = 0 \\ \partial_t E - \nabla \times B + \rho u - n v = 0 \\ \nabla \cdot E = \rho - n \\ \nabla \cdot B = 0 \ . \end{cases} \tag{1.2}$$

These systems are a basic physical model underlying several scientific branches that model electromagnetic effects such as plasma physics or semiconductor theory [4]. The densities and linear velocities of the fluids are denoted by ρ, n and u, v, respectively, while E, B stand for the electric and magnetic fields, respectively. Moreover, the functions p_1, p_2 represent the pressures and are given by $p_i(r) = r^{\gamma_i},\ r \geq 0,\ \gamma_i > 1,\ i = 1, 2.$

The purpose of this work is twofold. Firstly, one presents the relative energy identity for the bipolar Euler-Maxwell system (1.1)-(1.2); this being the content of Section 2. Particularly, in Section 3, one considers the bipolar Euler-Poisson system. Secondly, one gives two applications of the relative energy method in scenarios where the magnetic effects are neglected. The first application, presented in Section 4, concerns the weak-strong uniqueness property of a single-species Euler-Poisson system [1]. The second application, in Section 5, exposes the main result obtained in [2], which establishes the relaxation limit of a bipolar Euler-Poisson system with friction towards a bipolar drift-diffusion system.

The relative energy method is an efficient mathematical tool for stability analysis in the context of conservation laws; see [5] for early developments. It also proved to be useful in establishing limiting processes; see [9]. In what concerns mathematical literature on two-species Euler-Maxwell systems, refer to [3] for a formal derivation of two hierarchies of models from the bipolar Euler-Maxwell system with friction, and to [6] for a theory of existence of global smooth solutions to the bipolar Euler-Maxwell system.

2. Bipolar Euler-Maxwell system

Consider the following bipolar Euler-Maxwell system with dimensional constituents:

$$\begin{cases} \partial_t n_i + \nabla\cdot(n_i u_i) = 0 \\ \partial_t(m_i n_i u_i) + \nabla\cdot(m_i n_i u_i\otimes u_i) + \nabla p_i(n_i) = q_i n_i(E + u_i\times B) \\ \partial_t n_e + \nabla\cdot(n_e u_e) = 0 \\ \partial_t(m_e n_e u_e) + \nabla\cdot(m_e n_e u_e\otimes u_e) + \nabla p_e(n_e) = q_e n_e(E + u_e\times B) \\ \partial_t B + \nabla\times E = 0 \\ \varepsilon_0\partial_t E - \mu_0^{-1}\nabla\times B + q_i n_i u_i + q_e n_e u_e = 0 \\ \varepsilon_0\nabla\cdot E = q_i n_i + q_e n_e \\ \nabla\cdot B = 0 \end{cases} \tag{2.1}$$

For a species $j = i, e$ (ions and electrons), the density is denoted by n_j (m^{-3}), the linear velocity is represented by u_j ($\mathrm{m}\cdot\mathrm{s}^{-1}$), p_j ($\mathrm{kg}\cdot\mathrm{m}^{-1}\cdot\mathrm{s}^{-2}$) is the pressure, m_j (kg) is the mass and q_j ($\mathrm{s}\cdot\mathrm{A}$) is the charge. The electromagnetic field created by the charged particles is denoted by (E, B), where E ($\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-3}\cdot\mathrm{A}^{-1}$) stands for the electric field and B ($\mathrm{kg}\cdot\mathrm{s}^{-2}\cdot\mathrm{A}^{-1}$) for the magnetic field. The permittivity and permeability of free space are denoted by ε_0 ($\mathrm{kg}^{-1}\cdot\mathrm{m}^{-3}\cdot\mathrm{s}^{4}\cdot\mathrm{A}^{2}$) and μ_0 ($\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}\cdot\mathrm{A}^{-2}$), respectively. The identity $\varepsilon_0\mu_0 = 1/c^2$ holds, where c is the speed of light.

2.1. **Nondimensionalization.** Let e be the elementary charge and assume that $q_i = -q_e = e$. Following the ideas in [3, 7] one considers the scaling:

$$x = Lx' , \quad t = \tau_0 t'$$

$$n_j = N_0 n_j' , \quad u_j = v_0 u_j' , \quad v_0 = L/\tau_0 , \quad p_j = \kappa_B T_0 N_0 p_j' , \quad j = i, e$$

$$B = B_0 B' , \quad B_0 = \frac{\kappa_B T_0}{eLv_0} , \quad E = v_0 B_0 E' ,$$

where κ_B is the Boltzmann constant and T_0 is the temperature of the system. After dropping the primes and setting $\rho = n_i$, $n = n_e$, $u = u_i$, $v = v_e$, $p_1 = p_i$, $p_2 = p_e$, the non-dimensional version of

(2.1) becomes:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \zeta\big(\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u)\big) + \nabla p_1(\rho) = \rho(E + u \times B) \\ \partial_t n + \nabla \cdot (nv) = 0 \\ \epsilon\big(\partial_t(nv) + \nabla \cdot (nv \otimes v)\big) + \nabla p_2(n) = -n(E + v \times B) \\ \partial_t B + \nabla \times E = 0 \\ \lambda^2 \partial_t E - \beta^{-1} \nabla \times B + \rho u - nv = 0 \\ \lambda^2 \nabla \cdot E = \rho - n \\ \nabla \cdot B = 0 \end{cases} \tag{2.2}$$

where $\zeta = \frac{m_i v_0^2}{\kappa_B T_0}$, $\epsilon = \frac{m_e v_0^2}{\kappa_B T_0}$, $\beta = \frac{\mu_0 N_0 e^2 v_0^2 L^2}{\kappa_B T_0}$, $\lambda = \sqrt{\frac{\varepsilon_0 \kappa_B T_0}{e^2 L^2 N_0}}$. At this point one sets $\alpha = \frac{v_0^2}{c^2}$ and assumes that $\zeta = 1 \Leftrightarrow v_0 = \sqrt{\frac{\kappa_B T_0}{m_i}}$. Then $\beta = \frac{m_i N_0 v_0^2}{B_0^2/\mu_0} = \frac{N_0 \kappa_B T_0}{B_0^2/\mu_0}$ is the beta of the plasma, i.e. the ratio of the plasma pressure $N_0 \kappa_B T_0$ to the magnetic pressure B_0^2/μ_0, the scaled Debye length squared is given by $\lambda^2 = \frac{\alpha}{\beta}$, and $\epsilon = \frac{m_e}{m_i}$ is the quotient of the electron mass over the ion mass.

2.2. **Conserved quantities.** Here one presents the energy identity for system (2.2), which is formally derived by direct computation. First, one introduces the internal energy functions h_1, h_2 which are connected with the pressures p_1, p_2 through the following relations:

$$r h_i''(r) = p_i'(r) \ , \quad r h_i'(r) = p_i(r) + h_i(r) \ , \quad i = 1, 2 \ , \tag{2.3}$$

and are therefore given by

$$h_i(r) = \frac{1}{\gamma_i - 1} r^{\gamma_i} \ , \quad r \geq 0 \ , \quad \gamma_i > 1 \ , \quad i = 1, 2 \ .$$

In what follows $(\rho, \rho u, n, nv, E, B)$ is assumed to satisfy (2.2). Multiplying the first and second momentum equations of (2.2) by u and v, respectively, after using the remaining equations one obtains the energy identity of system (2.2):

$$\partial_t \mathcal{H} + \nabla \cdot \mathcal{F} = 0 \ , \tag{2.4}$$

where

$$\mathcal{H} = \zeta \tfrac{1}{2} \rho |u|^2 + \epsilon \tfrac{1}{2} n |v|^2 + h_1(\rho) + h_2(n) + \lambda^2 \tfrac{1}{2} |E|^2 + \beta^{-1} \tfrac{1}{2} |B|^2$$

is the total energy, and

$$\mathcal{F} = \zeta \tfrac{1}{2} \rho |u|^2 u + \epsilon \tfrac{1}{2} n |v|^2 v + \rho h_1'(\rho) u + n h_2'(n) v + \beta^{-1} E \times B$$

is the flux. Identity (2.4) represents the conservation of total energy of system (2.2).

In order to obtain another conserved quantity, one computes the evolution of the Poynting vector $E \times B$ and obtains

$$\partial_t \mathcal{M} + \nabla \cdot \mathcal{N} = 0 \ , \tag{2.5}$$

where

$$\mathcal{M} = \zeta \rho u + \epsilon n v + \lambda^2 E \times B \ ,$$

$$\mathcal{N} = \zeta \rho u \otimes u + \epsilon n v \otimes v + p_1(\rho) I + p_2(n) I + \lambda^2 \tfrac{1}{2} |E|^2 I + \beta^{-1} \tfrac{1}{2} |B|^2 I - \lambda^2 E \otimes E - \beta^{-1} B \otimes B \ .$$

2.3. **Relative energy identity.** Assume that $(\bar\rho, \bar\rho\bar u, \bar n, \bar n \bar v, \bar E, \bar B)$ also satisfies (2.2). The relative energy $\hat\Psi$ of system (2.2) is given by

$$\hat\Psi = \zeta \tfrac{1}{2} \rho |u - \bar u|^2 + \epsilon \tfrac{1}{2} n |v - \bar v|^2 + h_1(\rho|\bar\rho) + h_2(n|\bar n) + \lambda^2 \tfrac{1}{2} |E - \bar E|^2 + \beta^{-1} \tfrac{1}{2} |B - \bar B|^2 \ ,$$

where the relative quantity $h(r|\bar r)$ is given by $h(r|\bar r) = h(r) - h(\bar r) - h'(r)(r - \bar r)$.

Computing the time derivative of $\hat\Psi$ yields the relative energy identity of system (2.2):

$$\partial_t \hat\Psi + \nabla \cdot \hat\Theta = \hat\Sigma_1 + \hat\Sigma_2 + \hat\Sigma_3 + \hat\Sigma_4 \ , \tag{2.6}$$

where

$$\begin{aligned} \hat\Theta = & \ \zeta \tfrac{1}{2} \rho |u - \bar u|^2 u + \epsilon \tfrac{1}{2} n |v - \bar v|^2 v + h_1(\rho|\bar\rho) \bar u + h_2(n|\bar n) \bar v \\ & + \rho (h_1'(\rho) - h_1'(\bar\rho))(u - \bar u) + n (h_2'(n) - h_2'(\bar n))(v - \bar v) \\ & + \beta^{-1} (E - \bar E) \times (B - \bar B) \end{aligned}$$

is the relative flux, and

$$\begin{aligned} \hat\Sigma_1 &= -\zeta \nabla \bar u : \rho (u - \bar u) \otimes (u - \bar u) - \epsilon \nabla \bar v : n (v - \bar v) \otimes (v - \bar v) \ , \\ \hat\Sigma_2 &= -p_1(\rho|\bar\rho) \nabla \cdot \bar u - p_2(n|\bar n) \nabla \cdot \bar v \ , \\ \hat\Sigma_3 &= -((\rho - \bar\rho) \bar u - (n - \bar n) \bar v) \cdot (E - \bar E) \ , \\ \hat\Sigma_4 &= -(\bar u \times \rho (u - \bar u) - \bar v \times n (v - \bar v)) \cdot (B - \bar B) \ . \end{aligned}$$

Assuming that the internal energy functions h_1, h_2 are strictly convex, one can consider the relative energy $\hat\Psi$ as a way to compare two different states of system (2.2). As an illustration, assume that $(\rho, \rho u, n, n v, E, B)$, $(\bar\rho, \bar\rho\bar u, \bar n, \bar n \bar v, \bar E, \bar B)$ are two smooth and bounded away from vacuum solutions of (2.2) defined for times $t \in [0, T[$, and that $\gamma_1 = \gamma_2 \geq 2$. Integrating identity (2.6) over space gives

$$\frac{d}{dt} \Psi = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 \ ,$$

where $\Psi = \int \hat\Psi \ dx$ and $\Sigma_i = \int \hat\Sigma_i \ dx$, $i = 1, 2, 3, 4$. Then, one gets the bounds $\Sigma_i \leq C\Psi$, $i = 1, 2, 3, 4$, in the following way

$$\begin{aligned} \Sigma_1 &\leq C(||\nabla \bar u||_\infty + ||\nabla \bar v||_\infty) \int \zeta \frac{1}{2} \rho |u - \bar u|^2 + \epsilon \frac{1}{2} n |v - \bar v|^2 \ dx \leq C\Psi \ , \\ \Sigma_2 &\leq (||\nabla \cdot \bar u||_\infty + ||\nabla \cdot \bar v||_\infty) \int h_1(\rho|\bar\rho) + h_2(n|\bar n) \ dx \leq C\Psi \ , \\ \Sigma_3 &\leq C(||\bar u||_\infty + ||\bar v||_\infty) \int \frac{1}{2} h_1(\rho|\bar\rho) + \frac{1}{2} h_2(n|\bar n) + |E - \bar E|^2 \ dx \leq C\Psi \ , \\ \Sigma_4 &\leq (||\bar u||_\infty + ||\bar v||_\infty) \int \frac{1}{2} \rho |u - \bar u|^2 + \frac{1}{2} n |v - \bar v|^2 \ dx \\ & \quad + (||\bar u||_\infty + ||\bar v||_\infty)(||\rho||_\infty + ||n||_\infty) \int |B - \bar B|^2 \ dx \\ & \leq C\Psi \ . \end{aligned}$$

Thus,

$$\frac{d}{dt} \Psi \leq C\Psi \ ,$$

from which Gronwall's inequality yields the stability result $\Psi(t) \leq C(T)\Psi(0)$. Therefore, if the two different states coincide at the initial time, then they coincide at all times where they are defined.

3. Bipolar Euler-Poisson system

One would like to replicate the relative energy procedure detailed above for classes of solutions with less regularity. Some results on that direction have been obtained for particular cases where the magnetic effects are neglected. Taking $B = 0$ in (1.1)-(1.2) gives

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p_1(\rho) = \rho E \\ \partial_t n + \nabla \cdot (nv) = 0 \\ \partial_t(nv) + \nabla \cdot (nv \otimes v) + \nabla p_2(n) = -nE \\ \nabla \times E = 0 \\ \nabla \cdot E = \rho - n \ . \end{cases} \tag{3.1}$$

From the equation $\nabla \times E = 0$ one writes $E = -\nabla\phi$, where ϕ is the electric potential. Substituting this into system (3.1) one obtains the bipolar Euler-Poisson system, where the potential ϕ satisfies the Poisson equation $-\Delta\phi = \rho - n$. The solution of the Poisson equation is expressed as

$$\phi(t,x) = \big(K * (\rho - n)\big)(t,x) := \int_\Omega K(x,y)\big(\rho(t,y) - n(t,y)\big)dy \ ,$$

and its spatial gradient is understood as

$$\nabla\phi(t,x) = \big(\nabla_x K * (\rho - n)\big)(t,x) := \int_\Omega \nabla_x K(x,y)\big(\rho(t,y) - n(t,y)\big)dy \ ,$$

where the kernel K is either the Newtonian kernel in case $\Omega = \mathbb{R}^d$, or the Neumann function in case $\Omega \subseteq \mathbb{R}^d$ is a smooth bounded domain [8], $d \geq 3$. In either case, one has the bounds

$$|K(x,y)| \leq \frac{C}{|x-y|^{d-2}} \ , \quad |\nabla_x K(x,y)| \leq \frac{C}{|x-y|^{d-1}} \ . \tag{3.2}$$

3.1. **Riesz potentials and integration by parts formulas.** The theory of Riesz potentials proved to be useful to deal with the potential ϕ, essentially due to (3.2).

The Riesz potential of order $0 < \alpha < d$ of a function $f : \mathbb{R}^d \to \mathbb{R}$ is the function $I_\alpha f$ given by

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} \ dy \ .$$

If $f \in L^p(\mathbb{R}^d)$ for some $1 < p < d/\alpha$, then

$$||I_\alpha f||_{L^{\frac{dp}{d-\alpha p}}(\mathbb{R}^d)} \leq C||f||_{L^p(\mathbb{R}^d)} \ , \tag{3.3}$$

for some positive constant $C = C(\alpha, d, p)$ [10].

Now let $f = \rho - n$ so that $\phi = K * f$ and assume that $f \in L^1 \cap L^\gamma$. Combining (3.2) and (3.3) yields:

(1) if $\gamma \geq \frac{2d}{d+2}$ then $\phi \in L^{\frac{2d}{d-2}}$, $\nabla\phi \in L^2$,
(2) if $\gamma \geq \frac{2d}{d+1}$ then $f\nabla\phi \in L^1$.

With this at hand, using a density argument one derives two integration by parts formulas that will be crucial when dealing with the weak formulation of the system.

Let $g \in L^\gamma \cap L^1$ and $\varphi = K * g$. Then, if $\gamma \geq \frac{2d}{d+2}$ one has that

$$\int \nabla\phi \cdot \nabla\varphi \; dx = \int f\varphi \; dx = \int g\phi \; dx = \int\int f(x)K(x,y)g(y) \; dxdy \; . \tag{3.4}$$

Furthermore, if $\bar{u}$ is a smooth vector field vanishing at infinity and $\gamma \geq \frac{2d}{d+1}$, then

$$\int f\nabla\phi \cdot \bar{u} \; dx = \int \nabla\bar{u} : \nabla\phi \otimes \nabla\phi \; dx - \int (\nabla \cdot \bar{u})\tfrac{1}{2}|\nabla\phi|^2 \; dx \; . \tag{3.5}$$

4. Weak-strong uniqueness for the Euler-Poisson system

This section presents the weak-strong uniqueness principle for the following Euler-Poisson system in $]0,T[\times\mathbb{R}^d$, with $T < \infty$ and $d \in \mathbb{N}, \; d \geq 3$:

$$\begin{cases} \partial_t\rho + \nabla \cdot (\rho u) = 0 \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla\rho^\gamma + \rho\nabla\phi = 0 \\ -\Delta\phi = \rho \; . \end{cases} \tag{4.1}$$

In this case, the relative energy Ψ is given by

$$\Psi = \int_{\mathbb{R}^d} \tfrac{1}{2}\rho|u-\bar{u}|^2 + h\big(\rho|\bar{\rho}\big) + \tfrac{1}{2}|\nabla\big(\phi - \bar{\phi}\big)|^2 \; dx \; ,$$

and it is the tool used two compare two solutions. A stability identity satisfied by Ψ yields the following result:

Theorem 4.1. *Let $(\rho, \rho u, \phi)$ be a dissipative weak solution of (4.1) with $\gamma \geq \frac{2d}{d+1}$, and let $(\bar{\rho}, \bar{\rho}\bar{u}, \bar{\phi})$ be a strong solution of (4.1) with $\bar{\rho} > 0$. If $(\rho_0, \rho_0 u_0) = (\bar{\rho}_0, \bar{\rho}_0\bar{u}_0)$ then, for every $t \in [0,T[$,*

$$\big(\rho(t), \rho(t)u(t), \phi(t)\big) = \big(\bar{\rho}(t), \bar{\rho}(t)\bar{u}(t), \bar{\phi}(t)\big) \; .$$

A dissipative weak solution is a vector function $(\rho, \rho u)$ with $\rho \geq 0$ which satisfies (4.1) in a weak sense, has the properties of mass conservation and finitude of energy, and satisfies a weak form of the energy identity. By a strong solution $(\bar{\rho}, \bar{\rho}\bar{u})$ one understands a classical solution of (4.1) with bounded derivatives.

4.1. Sketch of the proof.

Let $(\rho, \rho u)$, with $\phi = \frac{1}{c(d)} I_2\rho$, be a dissipative weak solution of (4.1) with $\gamma \geq \frac{2d}{d+2}$, and let $(\bar{\rho}, \bar{\rho}\bar{u})$, with $\bar{\phi} = \frac{1}{c(d)} I_2\bar{\rho}$, be a strong solution of (4.1). Then, for each $t \in [0,T[$, the relative energy Ψ between these two solutions satisfies

$$\Psi(t) - \Psi(0) \leq \mathcal{J}_1(t) + \mathcal{J}_2(t) + \mathcal{J}_3(t) \; , \tag{4.2}$$

where

$$\mathcal{J}_1(t) = -\int_0^t \int_{\mathbb{R}^d} \nabla\bar{u} : \rho(u-\bar{u}) \otimes (u-\bar{u}) \; dxd\tau \; ,$$

$$\mathcal{J}_2(t) = -\int_0^t \int_{\mathbb{R}^d} (\nabla \cdot \bar{u}) p(\rho|\bar{\rho}) \; dxd\tau \; ,$$

$$\mathcal{J}_3(t) = \int_0^t \int_{\mathbb{R}^d} (\rho - \bar{\rho})\bar{u} \cdot \nabla(\phi - \bar{\phi}) \; dxd\tau \; .$$

Next, under the same conditions as Theorem 4.1 one obtains the bounds:

$$\mathcal{J}_i(t) \le C \int_0^t \Psi(\tau) d\tau \ , \quad t \in [0, T[\ , \quad i = 1, 2, 3 \ . \tag{4.3}$$

The bounds for the first two terms are obtained in a straightforward way, while the last term is handled using the integration by parts formula (3.5). Combining (4.2) with (4.3) gives

$$\Psi(t) \le \Psi(0) + C \int_0^t \Psi(\tau) \ d\tau \ , \quad t \in [0, T[\ ,$$

hence, by Gronwall's inequality,

$$\Psi(t) \le e^{CT} \Psi(0) \ , \quad t \in [0, T[\ ,$$

from which Theorem 4.1 follows by the strict convexity of $h(\rho) = \frac{1}{\gamma - 1} \rho^\gamma$ for $\rho > 0$.

5. Relaxation limit of the bipolar Euler-Poisson system

In this section one exposes the results obtained in [2]. It concerns the emergence of the bipolar drift-diffusion system

$$\begin{cases} \partial_t \rho = \nabla \cdot \big(\rho \nabla (h_1'(\rho) + \phi) \big) \\ \partial_t n = \nabla \cdot \big(n \nabla (h_2'(n) - \phi) \big) \\ -\Delta \phi = \rho - n \end{cases} \tag{5.1}$$

$$\rho \nabla (h_1'(\rho) + \phi) \cdot \nu = n \nabla (h_2'(n) - \phi) \cdot \nu = \frac{\partial \phi}{\partial \nu} = 0 \ , \quad \text{on } [0, T[\times \partial \Omega$$

as the relaxation limit of the bipolar Euler-Poisson system

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \frac{1}{\varepsilon} \nabla p_1(\rho) = -\frac{1}{\varepsilon} \rho \nabla \phi - \frac{1}{\varepsilon} \rho u \\ \partial_t n + \nabla \cdot (n v) = 0 \\ \partial_t (n v) + \nabla \cdot (n v \otimes v) + \frac{1}{\varepsilon} \nabla p_2(n) = \frac{1}{\varepsilon} n \nabla \phi - \frac{1}{\varepsilon} n v \\ -\Delta \phi = \rho - n \end{cases} \tag{5.2}$$

$$u \cdot \nu = v \cdot \nu = \frac{\partial \phi}{\partial \nu} = 0 \ , \quad \text{on } [0, T[\times \partial \Omega$$

in the space-time domain $]0, T[\times \Omega$, where $T > 0$ is a fixed time horizon and Ω is a smooth bounded domain of $\mathbb{R}^d$ with smooth boundary $\partial \Omega$, where $d \in \mathbb{N} \setminus \{1, 2\}$.

In this case, one regards a solution $(\bar\rho, \bar n, \bar\phi)$ of (5.1) as an approximate solution of (5.2) via

$$\begin{cases} \partial_t \bar\rho + \nabla \cdot (\bar\rho \bar u) = 0 \\ \partial_t (\bar\rho \bar u) + \nabla \cdot (\bar\rho \bar u \otimes \bar u) + \frac{1}{\varepsilon} \nabla p_1(\bar\rho) = -\frac{1}{\varepsilon} \bar\rho \nabla \bar\phi - \frac{1}{\varepsilon} \bar\rho \bar u + \bar e_1 \\ \partial_t \bar n + \nabla \cdot (\bar n \bar v) = 0 \\ \partial_t (\bar n \bar v) + \nabla \cdot (\bar n \bar v \otimes \bar v) + \frac{1}{\varepsilon} \nabla p_2(\bar n) = \frac{1}{\varepsilon} \bar n \nabla \bar\phi - \frac{1}{\varepsilon} \bar n \bar v + \bar e_2 \\ -\Delta \bar\phi = \bar\rho - \bar n \end{cases} \tag{5.3}$$

where

$$\bar{u} = -\nabla\big(h_1'(\bar\rho) + \bar\phi\big) \ , \quad \bar{v} = -\nabla\big(h_2'(\bar n) - \bar\phi\big) \ ,$$

$$\bar{e}_1 = \partial_t(\bar\rho\bar u) + \nabla\cdot(\bar\rho\bar u\otimes\bar u) \ , \quad \bar{e}_2 = \partial_t(\bar n\bar v) + \nabla\cdot(\bar n\bar v\otimes\bar v) \ ,$$

and compares the two solutions using the relative energy Ψ given by

$$\Psi = \int_\Omega \varepsilon\tfrac{1}{2}\rho|u-\bar u|^2 + \varepsilon\tfrac{1}{2}n|v-\bar v|^2 + h_1(\rho|\bar\rho) + h_2(n|\bar n) + \tfrac{1}{2}|\nabla(\phi-\bar\phi)|^2 \, dx \ .$$

The following result is obtained:

Theorem 5.1. *Let $(\rho, \rho u, n, nv)$, with $\phi = N*(\rho-n)$, be a dissipative weak solution of (5.2) with $\gamma_1,\gamma_2 \geq 2-\frac{1}{d}$, and let $(\bar\rho,\bar\rho\bar u,\bar n,\bar n\bar v)$, with $\bar\phi = N*(\bar\rho-\bar n)$, be a strong and bounded away from vacuum solution of* (5.1)*. There exists $C>0$ such that for $t\in[0,T[$ the relative energy Ψ between these two solutions satisfies the stability estimate*

$$\Psi(t) \leq e^{CT}\big(\Psi(0)+\varepsilon^2\big) \ . \tag{5.4}$$

Therefore if $\Psi(0)\to 0$ as $\varepsilon\to 0$, then $\Psi(t)\to 0$ as $\varepsilon\to 0$ for every $t\in[0,T[$.

By a dissipative weak solution one understands a vector function $(\rho,\rho u,n,nv)$, with $\phi = N*(\rho-n)$, that satisfies (5.2) in a weak sense, has finite energy and conserved mass, and satisfies a weak form of the energy identity. Strong solutions $(\bar\rho,\bar\rho\bar u,\bar n,\bar n\bar v)$, $\bar\phi = N*(\bar\rho-\bar n)$, of (5.1) are classical solutions with bounded derivatives. Similarly as in the proof of Theorem 4.1, the first step towards the conclusion of Theorem 5.1 is a relative energy inequality satisfied by a weak solution of (5.2) and a strong solution of (5.1). Then, one bounds the terms on the right-hand side of the relative energy inequality in terms of Ψ and concludes using Gronwall's inequality. The term on the right-hand side of the relative energy inequality that deserves more attention is the one containing the electric potentials. To handle this term one uses the estimate (3.3) together with interpolation of L^p spaces. For the full details refer to [2].

References

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